JHEP HYPER VERSION

RECEIVED: *October 23, 1998*, ACCEPTED: *December 8, 1998*

U-duality of Born-Infeld on the noncommutative two-torus

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ABSTRACT: We discuss Born-Infeld on the noncommutative two-torus as a description of compactified string theory. We show that the resulting theory, including the fluctuations, is manifestly invariant under the T-duality group $\mathrm{SO}(2,2;\mathbb{Z})$. The BPS mass even has a full $\mathrm{SL}(3,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$ U-duality symmetry. The direct identification of the noncommutative parameter θ with the B-field modulus however seems to be problematic at finite volume.

KEYWORDS: String Duality, D-Branes, Noncommutative Torus.

JHEP12(1998)010

Contents

1. Introduction

String theory compactified on a torus is expected to be invariant under a discrete group of U-dualities. An $\mathrm{SO}(n,n;\mathbb{Z})$ part of this duality group turns up as T-duality. This duality is already manifest in the perturbative description of string theory. To see the full duality group, also non-perturbative information about string theory should be considered. Especially D-branes, which turn up as non-perturbative objects in string theory, give necessary states related by U-duality to the perturbative string states. In the well-known description of these objects in terms of Dirichlet boundary conditions for (perturbative) strings, the T-duality transformations remain valid, and act nontrivially on these objects. Already quite some time ago it was found that at low energies the world-volume theory of these objects should be described by a Born-Infeld gauge theory [1]. A gauge theory interpretation also turns up when one looks at the M(atrix) theory description of D0-branes [2, 3]. There it was found that the momentum modes of D-particles describe a gauge theory on the torus dual to the one on which these particles live. In this world-volume theory, even the T-dualities should act quite nontrivial. For example, for compactification on the two-torus there is a duality which interchanges the rank of the gauge group with the total magnetic flux. This is a two-dimensional version of Nahm duality.

The dualities act nontrivial on al the moduli of the string theory. One of these moduli is the B-field. Recently it was found that when a nonzero B-flux is turned on in string theory, the gauge theory describing the D-particles should live on a noncommutative torus. this means that the coordinates x^i of the (dual) torus where the gauge theory lives should satisfy nontrivial commutation relations (already at the classical level)

$$[x^i, x^j] = \frac{\theta^{ij}}{2\pi i}\,. \tag{1.1}$$

It has been argued that the parameter θ should be identified with a constant B-field background. This was first discussed in the context of matrix theory in [4], and discussed thereafter in many papers, see e.g. [5]–[15]. As M(atrix) theory, or more correctly DLCQ of M-theory compactified on a torus, effectively describes the dynamics of D-branes, this means that also the gauge theory on the worldvolume of these branes should be described by a theory on the noncommutative torus.

Combining these known fact, we are led to conjecture that the low energy theory on the worldvolume of the D-branes is described by Born-Infeld on a noncommutative torus. In this paper we shall consider this proposal for the compactification of string theory on a two-torus. This means that we are describing the worldvolume theory on the D2-brane. The description using noncommutative geometry has a definite advantage above the description using classical geometry. The reason for this is that there is a manifest equivalence between gauge bundles over noncommutative tori, called Morita equivalence. This equivalence, which actually is an identification of adjoint bundles on the torus, has been argued to be the manifestation of T-duality in the gauge theory description. In particular, it includes the above mentioned Nahm duality.

In the next subsection, we shortly discuss some well known facts about systems of D0-brane on a two-torus and dualities expected from string theory. It will be mainly used to set some notations.

In section 2 we describe the 'classical' Born-Infeld theory, which should describe these systems, at least for a single D0-brane, in a background with vanishing Kalb-Ramond field. We will then discuss a direct generalization of this theory for the non-abelian situation with $B \neq 0$, in terms of a local field theory.

In section 3, a generalization of the Born-Infeld theory living on the noncommutative torus will be introduced. The Hamiltonian of this theory will be fixed by requiring to have the same dependence on the global zero-modes, as this was already correctly described in the naive generalization. We discuss a *manifest* invariance of the Hamiltonian under the full T-duality group $SO(2,2;\mathbb{Z})$. We also consider the Yang-Mills limit of the Born-Infeld theory.

In section 4 we give an overview of the noncommutative torus and gauge bundles on this geometry. This section will be more technical. The description will be quite explicit, in order to make it understandable without much prerequired knowledge.

Section 5 concludes with some discussion on what we found.

1.1 D0-branes on $\mathbb{T}^2$ and duality

In this paper, we are interested in the dynamics of D0-branes in IIA string theory compactified on a two-dimensional torus $\mathbb{T}^2$. Apart from the D0-branes, we also allow for wrapped D2-branes, string winding and momentum. The system can be looked at from various points of view. The interpretation of the charges and moduli depend on the perspective one takes. From the string point of view, we distinguish the T-dual D0- and D2-brane picture. In the two tables 1 and 2 we compare the various charges and moduli in these pictures.

charge	D0-brane	D2-brane	Yang-Mills
N	D0	D2	rank
M	D2	D0	$\int \mathrm{Tr}\, F$
n^i	mom.	F1	$\int \mathrm{Tr}\, E^i$
m_i	F1	mom.	$\int \mathrm{Tr}\, P_i$

Table 1: Charges in the various pictures.

modulus	D0-brane	D2-brane
metric	g_{ij}	G_{ij}
B-field	b	B
coupling	λ_s	λ'_s
RR-field	c_i	C_i

Table 2: Moduli in these pictures.

The D2-branes are described by a non-abelian three-dimensional Born-Infeld theory. The various quantum numbers in this gauge theory, rank, electric fluxes, magnetic flux and momentum, are identified with the charges in the string compactifications, as given in table 1.

On the charges and moduli act first of all the T-duality group $SO(2,2;\mathbb{Z}) = SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$. Note that the moduli in the D0- and D2-brane picture are related by a T-duality transformation. The T-duality group breaks up into two $SL(2,\mathbb{Z})$ groups. One of them is the mapping class group of the torus, which acts trivially on the gauge theory. The other factor acts quite nontrivial. It relates the D0-brane picture and the D2-brane picture. This factor will be called $SL(2,\mathbb{Z})_N$ in the following. One of the generators of this group acts by Nahm transformations on the gauge theory. This duality will be our main interest in this paper.

To make the action of T-duality on the moduli more obvious, we express the NS-moduli in terms of complex moduli. To be precise, we define a shape parameter τ and a complexified Kähler modulus σ by

$$\tau = \frac{g_{12} + i\sqrt{\det g}}{g_{11}}, \qquad \sigma = b + i\sqrt{\det g}\,, \tag{1.2}$$

The B-field modulus B and the volume $\sqrt{\det G}$ in the D2-brane picture are related to the latter modulus as $B+i\sqrt{\det G} = -\sigma^{-1}$. The RR-forms in the two pictures are related by Hodge-duality, $c_i = {}^*C_i = -\epsilon^{ij}C_j$. The string coupling is not invariant under T-duality. The invariant coupling is the effective coupling in the eight-dimensional transverse world, given by $\lambda_8 = \lambda_s/(\det g)^{1/4} = \lambda'_s/(\det G)^{1/4}$.

In the full M-theory, the T-duality group is enhanced to the U-duality group $\mathrm{E}_{3(3)}(\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(3,\mathbb{Z})$. As Type IIA string theory can be seen as M-theory compactified on a circle, the compactification under consideration is M-theory compactified on a three-torus. This explains the second factor of the U-duality group, as it is the enhanced mapping class group of the torus. The first factor is hence the $\mathrm{SL}(2,\mathbb{Z})_N$ duality group. The charges in table 1 transform in the $(\mathbf{1},\mathbf{2}) \oplus (\mathbf{2},\mathbf{2})$ representation of the T-duality group. With the enhancement of the duality group, the charges combine to a $(\mathbf{3},\mathbf{2})$ representation of the U-duality group. The moduli (including the string coupling) parametrize the moduli space

$$\mathcal{M}_3 = \frac{\mathrm{SL}(2)}{\mathrm{SO}(2)} \times \frac{\mathrm{SL}(3)}{\mathrm{SO}(3)}\,, \tag{1.3}$$

where the U-duality should also be divided out. This moduli space can be described by the $\mathrm{SL}(2)_N$ modulus σ, and a three-dimensional metric $\widehat{G}_{\hat{\imath}\hat{\jmath}}$, build from the remaining moduli τ, λ_8 and c_i. This metric is given by

$$\widehat{G}_{\hat{\imath}\hat{\jmath}} = \frac{\ell_{pl}^2}{(\lambda_s^2 \det g)^{1/3}} \begin{pmatrix} g_{ij} + \lambda_s^2\, c_i c_j & \lambda_s^2\, c_i \\ \lambda_s^2\, c_j & \lambda_s^2 \end{pmatrix}, \tag{1.4}$$

where we introduced the eight-dimensional Planck length $\ell_{pl} = \lambda_8^{1/3}\ell_s$. This length unit is invariant under all U-duality transformations, in contrast to the string length. Therefore it is appropriate to express all dimensionful quantities that are supposed to transform covariantly under the U-duality group in terms of this unit. The three-dimensional metric in (1.4) is not the natural SUGRA metric of M-theory. It is rescaled by a factor $\sqrt{\det g}$, which equals the three-dimensional volume of the torus on which the M-theory is compactified (in eleven-dimensional Planck units), the only parameter available in the M-theory compactification. Note that the factor is such that the metric is independent of σ, so that the moduli space is factorized.

2. Abelian Born-Infeld on the classical torus

In this section we study the Born-Infeld description on the 'classical' two-torus of Type IIA D-branes. At least for the abelian situation arising on single two-branes, and with vanishing Kalb-Ramond field, this should be the correct description of the effective theory. Starting from this action we shall obtain the full BPS spectrum expected from string theory from the Yang-Mills theory, including all the moduli.

2.1 Lagrangian and Hamiltonian

The correct effective description of D-branes at zero Kalb-Ramond field has been known for quite some time [1], and is described by a Born-Infeld action. This action for the abelian gauge theory is given by

$$S_{BI} = \int d^2x dt \, \frac{-1}{\lambda_s'} \sqrt{\det(-G - F + B)} + C \wedge F \,. \tag{2.1}$$

Here we used units such that the string length is $\ell_s = 1$. The appearance of the inverse power of the ten-dimensional string coupling is natural for the RR-soliton, as a boundary for fundamental strings. The last term is the coupling to the RR one-form field. There are only fluxes in the two spatial torus directions, as these are the only compact directions. Hence this term involves only the electric components F_{0i} of the field strength. We included the B-field as a shift of the field strength, as in [16].

Our next step is to calculate the Hamiltonian associated to the action. This is expressed in terms of the field strength, the electric field and the momentum density, given by

$$F = \partial_1 \mathcal{A}_2 - \partial_2 \mathcal{A}_1 \,, \qquad E^i = \frac{\delta \mathcal{L}}{\delta \dot{\mathcal{A}}_i} \,, \qquad P_i = F_{ij} E^j . \tag{2.2}$$

The zero-modes of these operators are the various quantum numbers, associated to the different branes, windings and momentum, as given in table 1. We normalize these operators such that the zero-modes are integral.

Straightforward calculation of the Hamiltonian density then yields the result

$$\begin{aligned} \mathcal{H} = E^i \dot{\mathcal{A}}_i - \mathcal{L}_{BI} = \frac{1}{\ell_s} \bigg(& \frac{1}{\lambda_8^2 \sigma_2} |1 + \sigma F|^2 + \frac{1}{\sigma_2 \tau_2} \Big| -\bar{\tau} E^1 + E^2 + \\ & + \sigma(P_1 + \bar{\tau} P_2) + (C_1 + \bar{\tau} C_2)(1 + \sigma F) \Big|^2 \bigg)^{1/2} , \end{aligned} \tag{2.3}$$

where in the last line we reintroduced units.

We now combine the various operators into an 'electric' and 'magnetic' three-vector. This allows us to write the Hamiltonian density in a form where the duality transformations are much more obvious, at least as far as the zero-modes are concerned. These vectors are

$$\mathcal{E}_{\hat{i}} = \begin{pmatrix} E^i \\ I \end{pmatrix} , \qquad \mathcal{B}_{\hat{i}} = \begin{pmatrix} -\epsilon^{ij} P_j \\ F \end{pmatrix} . \tag{2.4}$$

Norms for these vectors should be calculated using the three-dimensional metric (1.4).

Note the position of the indices, which are different because we are comparing matrices in T-dual pictures: the objects $\mathcal{E}$ and $\mathcal{B}$ should be interpreted in the context of the three-dimensional enhancement, which naturally arises starting from the D0-brane picture. Using these three-dimensional quantities, including the three-dimensional metric (1.4), we can write the Hamiltonian density in the suggestive form

$$\mathcal{H} = \frac{\|\mathcal{E} + \sigma\mathcal{B}\|}{\ell_{pl}\sqrt{\sigma_2}} \,. \tag{2.5}$$

Here $\mathcal{E}$ and $\mathcal{B}$ have lower indices, hence we used the inverse of the metric (1.4) (with upper indices) to calculate the norm.

From an M-theory perspective, the above way of writing the Hamiltonian is very natural. This is because in M-theory the system is compactified on a three-torus. The vectors $\mathcal{E}$ and $\mathcal{B}$ are related to the momentum density and membrane-wrapping respectively ($\mathcal{B}$ is related to a two-vector). In this interpretation, the lower indices are quite natural.

2.2 Naive generalization and BPS spectrum

The result for the Hamiltonian density in the last subsection suggests a simple generalization for the non-abelian theory, by replacing the operators in $\mathcal{E}$ and $\mathcal{B}$ by their non-abelian counterparts and putting a suitable trace in front of the expression (2.5). It was suggested in [17] that this trace should be the symmetric trace. Anyway, as far as the zero-modes and the BPS masses are concerned, the precise form of the trace is not so relevant. The zero-modes of the local operators are given by

$$\mathcal{N} = \int d^2x\, \mathrm{Tr}\, \mathcal{E} = \begin{pmatrix} n^i \\ N \end{pmatrix} , \qquad \mathcal{M} = \int d^2x\, \mathrm{Tr}\, \mathcal{B} = \begin{pmatrix} -\epsilon^{ij} m_j \\ M \end{pmatrix} . \tag{2.6}$$

Inserting these zero-modes in the Hamiltonian (2.5), we see that this gives the correct zero-mode contribution, invariant under the full U-duality group $\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(3,\mathbb{Z})$. The BPS mass levels can be calculated as a lower bound on the energy. They come out correctly in a manifest U-duality invariant way, as they are given by

$$\ell_{pl}^2 M_{BPS}^2 = \frac{\|\mathcal{N} + \sigma\mathcal{M}\|^2}{\sigma_2} + 2\|\mathcal{N} \times \mathcal{M}\| \,. \tag{2.7}$$

This result can be derived methods similar to that used in [18] for string compactification on the four-torus. In fact, it is a dimensional reduction to two dimensions of this result. It equals the direct generalization for $b \neq 0$ of the result of [19] reduced to two dimensions, which was calculated using the supersymmetry algebra. Note that this result is certainly correct for $N = 1$ and $b = 0$, as we know that then the Born-Infeld theory is correct. For different values of N and b, this expression is the unique expression for the BPS mass which is invariant under the full expected U-duality. Therefore, whatever theory should describe the D0-branes, we know that it must always give rise to these BPS masses. In the full supersymmetric Born-Infeld theory, these BPS states are states that break half or a quarter of the supersymmetry. For the 1/2 BPS states, the mass was already calculated in [20, 21, 22, 23] (even for compactification on the four-torus). These states correspond to states with $\mathcal{N} \times \mathcal{M} = 0$, therefore the fluctuations do not contribute to the BPS mass. A nice and recent review of the subject of U-duality and BPS-formulae in M-theory can be found in [24].

2.3 Towards duality invariance

Looking at the Hamiltonian density (2.5), duality invariance seems quite straightforward. Of particular interest in this paper is the $\mathrm{SL}(2,\mathbb{Z})_N$ duality under which the modulus σ transforms. Let us look at this duality transformation in some more detail. The form of the Hamiltonian (2.5) suggests that the two vectors $\mathcal{E}$ and $\mathcal{B}$ transform more or less as a doublet under this $\mathrm{SL}(2,\mathbb{Z})_N$. To guarantee invariance under this group of at least the contribution from the zero-mode to the Hamiltonian; the zero-modes should transform exactly as a doublet. This implies that the full operators $\mathcal{E}$ and $\mathcal{B}$ transform as a doublet up to normalization. The normalization can be found from the transformation of the rank $N = \mathrm{Tr}\, I$.

This simple rule is certainly true at the level of zero-modes. For the complete fields including the fluctuations however, this can not be the full story. To see this, remark that the unit operator I is interchanged with the magnetic field F under the transformation mentioned above. The latter operator however has fluctuations, while the unit operator certainly has no fluctuations. Furthermore, the unit operator should somehow be the unit operator in all representations. So if we want to view duality as some map between different representations, we expect that this operator at least should remain intact. And this certainly is not true for this simple transformation rule.

The problem noted above presents itself quite prominent in the metric for the fluctuations. To see this, we write down the quadratic action for the fluctuation of the magnetic field. For simplicity, and also to make comparison to the situation on the noncommutative torus later on, we consider the small volume limit $g \to 0$. We split the magnetic field into a zero-mode part and a fluctuation part, $F = \frac{M}{N} I + F'$. Then by expanding around the zero-modes, we find the following quadratic action for the fluctuation part F'

$$S_{F'} = \frac{N\sigma_2^{3/2}}{\lambda_8 \ell_{pl} |N + Mb|} \int d^2x dt \ \mathrm{Tr}(F')^2 \,. \tag{2.8}$$

Let us now see how this transforms under the T-duality transformation

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})_N \,. \tag{2.9}$$

From the transformation of the zero-modes, we read off that F' has to transform according to $NF' \to DNF'$. Furthermore, Tr transforms as $N = \mathrm{Tr}\, I$. Also we know that the volume v scales in the small volume limit according to

$$\sigma_2 \to \frac{\sigma_2}{(Cb+D)^2} \,. \tag{2.10}$$

In the small volume limit the zero-mode combination $|N+Mb|/\sqrt{\sigma_2}$ is invariant. Combining these transformations, we find that the quadratic fluctuation part (2.8) transforms with a factor $\frac{D^2}{(Cb+D)^2}$, which is different from unity. Therefore, the metric for the fluctuations is not invariant. Note that this problem is not a consequence of the generalization to a non-abelian gauge theory, because it already happens for transformations that leave us in the abelian situation $N=1$. We shall see below how this changes on the noncommutative torus.

3. Born-Infeld on the noncommutative torus

In the last section we found that the non-abelian generalization of the Hamiltonian density given in (2.5) gives rise to the correct BPS masses, and is U-duality invariant at the zero-mode level. As we saw however, at the local level invariance breaks down when b becomes nonzero. The fact that we successfully reproduced the correct BPS masses indicates that the form of the Hamiltonian density can still be true, and must be correct for the global modes. We might however try and change the identification of the local operators in this formula, to correct for the flaws of the theory. This means that we should change the identification of the local forms of $\mathcal{E}$ and $\mathcal{B}$, but in such a way that they still have the same (integral) zero-modes. At first sight this seems impossible. But here the recent relation of string compactifications at nonzero b with noncommutative geometry presents a different solution, which, as we shall see in the rest of the paper, will give a solution to this problem.

3.1 Hamiltonian of Born-Infeld on the noncommutative torus

Motivated by the above mentioned observations, we now consider Born-Infeld on the noncommutative torus $\mathbb{T}^2_\theta$. As one consequence of this, the fluxes of the various operators will change. As noted above, this means that we should replace the vectors $\mathcal{E}$ and $\mathcal{B}$ of local operators by different ones. These replacements should be chosen such that they have the same zero-modes, as we do trust the zero-mode contribution to the Hamiltonian. First let us introduce the generalizations of the three-vectors of local operators in (2.4) for the noncommutative torus, and denote them $\mathcal{E}_\theta$ and $\mathcal{B}_\theta$. So e.g. $\mathcal{E}_\theta = (E^i, I)$. On the noncommutative torus, the zero-modes of the various local operators are not integral. They are in general shifted, and are given by the formulae (4.11) and (4.18); this will be discuss this in more detail later. Denoting the zero modes of $\mathcal{E}_\theta$ and $\mathcal{B}_\theta$ by $\mathcal{N}_\theta$ and $\mathcal{M}_\theta$ respectively, these shifts can be summarized as

$$\int d^2x \, \mathrm{Tr}_\theta \, \mathcal{E}_\theta \equiv \mathcal{N}_\theta = \mathcal{N} + \theta\mathcal{M}, \qquad \int d^2x \, \mathrm{Tr}_\theta \, \mathcal{B}_\theta \equiv \mathcal{M}_\theta = \mathcal{M} \,, \tag{3.1}$$

where Tr_θ denotes the trace for the bundle on the noncommutative torus. These shifts enforce us to identify the three-vectors $\mathcal{E}$ and $\mathcal{B}$ appearing in the Hamiltonian as shifted fields according to

$$\mathcal{E} = \mathcal{E}_\theta - \theta \mathcal{B}_\theta, \qquad \mathcal{B} = \mathcal{B}_\theta \,, \tag{3.2}$$

in order for them to have integer zero-modes. Inserting these values, we find that the Hamiltonian (2.5) on the noncommutative torus $\mathbb{T}^2_\theta$ can be written

$$\mathcal{H} = \frac{\mathrm{Tr}_\theta \, \|\mathcal{E}_\theta + (\sigma - \theta)\mathcal{B}_\theta\|}{\ell_{pl}\sqrt{\sigma_2}} \,. \tag{3.3}$$

By construction, the zero-mode contribution is exactly the same as for the theory on the classical torus. Careful inspection of the derivation shows that even the BPS mass spectrum is the same. This also means that this BPS spectrum is invariant under the T-duality group. As we shall see below, we have even more.

3.2 Manifest $\mathrm{SL}(2,\mathbb{Z})_N$ duality

Now that we have constructed the theory on the noncommutative torus, we can start to analyze it. We already saw that we have the correct BPS mass levels. Also the $\mathrm{SL}(2,\mathbb{Z})$ part of the T-duality group arising from the mapping class group of the torus is manifestly present on the noncommutative torus. The nontrivial part to check is the $\mathrm{SL}(2,\mathbb{Z})_N$ part, which acts on the modulus σ.

As we already noted before, the local operators $(\mathcal{E}, -\mathcal{B})$ should transform, modulo normalization, as a doublet under this duality group, because the zero-modes of these operators do so. In order to find the correct transformation of the zero-modes, the local operators should transform according to the rule

$$\begin{pmatrix} \mathcal{E}' \\ -\mathcal{B}' \end{pmatrix} = d \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathcal{E} \\ -\mathcal{B} \end{pmatrix}, \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})_N \,, \tag{3.4}$$

where $d = \mathrm{Tr}_\theta \, I / \mathrm{Tr}_{\theta'} \, I$ takes care of the difference in normalization. In noncommutative geometry, it has the interpretation of the scaling of an abstract dimension of the fiber of the bundle over the noncommutative torus (which is not necessarily integral on the noncommutative torus). To appreciate this, note that for the classical torus it equals the scaling of the rank N.

Now let us see what this means for the fields on the noncommutative torus. Before we noted that problems arose because the unit operator transforms into a combination of the unit operator and the field strength. So let us try to find a transformation such that the transformation of the unit operator does not get additional contributions. It turns out that, with the transformation rule (3.4) and the identifications (3.2), the noncommutative parameter θ must transform with fractional linear transformations

$$\theta' = \frac{A\theta + B}{C\theta + D} \,. \tag{3.5}$$

Using this, we can calculate the scaling of the trace from the formula for the trace of the unit

$$d = \frac{N + M\theta}{N' + M'\theta'} = C\theta + D\,. \tag{3.6}$$

With the above transformation of θ, the transformation (3.4) of the shifted fields can be rewritten in terms of the unshifted variables, related as in (3.2), as

$$\begin{pmatrix} \mathcal{E}'_{\theta'} \\ -\mathcal{B}'_{\theta'} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ C(C\theta + D) & (C\theta + D)^2 \end{pmatrix} \begin{pmatrix} \mathcal{E}_{\theta} \\ -\mathcal{B}_{\theta} \end{pmatrix}. \tag{3.7}$$

Of particular interest is the zero in the upper right corner of this transformation matrix, which assures that the unit operator, which is a component of $\mathcal{E}_\theta$, does not get a contribution from $\mathcal{B}_\theta$. (The unit operator is even invariant, as the upper left element is equal to one. This is a direct consequence of our definition of d). The above transformation rules can now be consistently written in terms of transformations of the independent fields F and E^i according to

$$F' = (C\theta + D)^2 F - C(C\theta + D) I\,, \qquad (E^i)' = E^i\,. \tag{3.8}$$

As we shall see, the transformation of the field strength F can consistently be written in terms of transformation of the gauge field $\mathcal{A}_i$. These are exactly the transformations under Morita-equivalence, which we discuss later. So we find that in our Born-Infeld theory on the noncommutative torus, T-duality can be described by Morita equivalence.

3.3 Relation to string theory

We shall now discuss the relations of the above description of Born-Infeld theory on the noncommutative torus with string compactifications. A direct identification of this theory as the effective world-volume theory on the D2-brane runs into an obvious problem: there are too many parameters. The string theory has the 7 moduli g_{ij}, b, c_i and λ_s, while the theory on the noncommutative torus has the additional parameter θ. It was argued in several papers, starting from [4], that the noncommutative parameter θ should be identified with the B-field modulus b. This has the interesting implication that in the Hamiltonian (3.3) the explicit dependence on b drops out; the Hamiltonian only depends on b via the noncommutativity encoded in θ.

This is not the complete solution however. The transformation of θ and b that we derived are different from each other, as b transforms with fractional linear transformations together with $\det g$, in the combination $\sigma = b + i\sqrt{\det g}$. This has the nasty implication that if we identify θ with b in some formulation, then after doing an $\mathrm{SL}(2,\mathbb{Z})_N$ duality rotation, the two parameters are not identified any more. There can be two ways out of this dilemma. The first is the possible appearance of a continuous symmetry which allows us to change θ to any value; so we can use this to just set θ equal to b. This would mean that they are not independent parameters, but rather their sum should be identified with the B-field. The other way out is a deformation of the noncommutative geometry to a torus geometry depending on the a complexified modulus $\theta \in \mathbb{C}$. We can then identify θ with the full complex modulus σ. We believe that the second one is the most plausible. This is because we see no reason for the continuous symmetry which otherwise is necessary to exist. Also, the generalization of the real parameter θ in noncommutative geometry to a complex one is quite natural, as we know that there is an $\mathrm{SL}(2,\mathbb{Z})$ duality group acting on this parameter; and in most cases where this happens, this parameter can in fact be complexified. On the other hand, we do not know such a generalization to exist. This solution implies that the description of the noncommutative torus used here as a description of string theory compactifications is only correct when σ and θ are identified, hence in the small volume limit $g \to 0$. If this conjecture is true, the identification of θ with b is only valid in the small volume limit $\sigma_2 = 0$. This is something that, in principle, could be argued from the string theory by a refinement of the arguments in the literature that lead to the appearance of noncommutative geometry in string compactifications.

3.4 Lagrangian and Yang-Mills limit

We now study the Lagrangian and the Yang-Mills limit of the Born-Infeld theory on the noncommutative torus. We will assume here that θ is identified with the B-field modulus b. The Hamiltonian density on the noncommutative torus that we found can easily be derived from a classical Lagrangian. Noting that the form of the Hamiltonian density (3.3) is similar to (2.5) with the explicit b-dependence taken out, the action from which it should be derived is

$$S_{BI} = \int d^2x dt \ \mathrm{Tr}_\theta \left(\frac{-1}{g_{YM}^2} \sqrt{\det(-g^{-1} - F)} + C \wedge F \right) . \tag{3.9}$$

The coupling that appears in this action should be $g_{YM}^2 = \lambda_8/(\det g)^{1/4} = \lambda_s/\sqrt{\det g}$, to match the coupling in the Hamiltonian. Note that this action in the limit of small coupling, which is the limit where noncommutative geometry should appear, reduces to a normal Yang-Mills action. The coupling of this Yang-Mills theory is given by g_{YM}.

How can we interpret this result for the action? We see that the metric on the noncommutative torus is the inverse metric to the D0-brane metric g, and not the T-dual metric G. This can easily be understood, if we remember that the gauge theory arises as the effective theory of the D0-branes represented in terms of the momentum modes. Therefore the metric should be the metric on the dual torus where the momenta live, and this is the inverse metric.

Another hint comes from the transformation of the field strength (3.8). The factor $C\theta + D$ is exactly the scaling of the volume $\sqrt{\det g^{-1}}$ under T-duality in the small volume limit $g \to 0$, and therefore it is the conformal scaling of the metric g^{-1}. This factor can therefore be absorbed by a scaling of the coordinates.

We shall now consider the fluctuation part in the Yang-Mills limit in some more detail. The form of the Hamiltonian shows clearly that this limit arises when we take both g and λ_8 to zero. That is, we take the small coupling and small volume limit. The Hamiltonian then takes the quadratic form

$$H = H_0 + \frac{1}{\ell_{pl}\lambda_8\sqrt{\sigma_2}} \int d^2x dt \; \mathrm{Tr}_\theta \left(\frac{\lambda_8^2}{2}(E')^2 + \frac{\sigma_2^2}{2}(F')^2 \right) , \tag{3.10}$$

where the primes indicate that we omit the zero-modes of the operators. The first part, $\ell_{pl} H_0 = \sigma_2^{-1/2} \|\mathcal{N} + \sigma \mathcal{M}\|$, is the zero-mode contribution, which is invariant under the complete U-duality group. This fluctuation part is now invariant under Morita equivalence or T-duality in the small volume limit, as F' scales with $(C\theta + D)^2$, while σ_2 scales with the inverse factor. The overall factor $\sqrt{\sigma_2}^{-1}$ absorbs the scaling of the trace. The coupling λ_8 and the electric field E' are both invariant.

As the zero mode part is invariant under the T-duality transformations, it is not completely invariant under the small g version, which we saw was equivalent to Morita equivalence. Any transformation in this part can however be undone by changing the value of certain components of the three-form field, which turn up in the Lagrangian by uninteresting constant terms like $\int \mathrm{Tr}_\theta I$ and $\int \mathrm{Tr}_\theta F$. This was the point of view taken in [4] and other papers following it. We do however take the point of view that not these parameters should be changed, but the transformation rules of the fields should be changed to the ones predicted by T-duality. As far as the zero-modes are concerned, we already saw that the Hamiltonian is then exactly invariant, at least if we take the full Born-Infeld theory. For the fluctuation part, we do not know how to correct the transformation rules. Indeed, as Morita equivalence is an exact duality for the bundles on the noncommutative torus and not the full duality involving σ_2, the basic definitions of this torus geometry should be changed. Somehow, the real parameter b should be generalized to σ in the complex upper half-plane.

There is a reason for taking the small volume limit, which is the decoupling of stringy corrections[1]. For this, we need to take α' (or ℓ_s) to zero. Doing this, we should be carefull to keep the volume from the gauge theory point of view finite. Otherwise, we can not really make sense out of the gauge theory. This volume is the dimensionfull volume $\ell_s^2\sqrt{\det g^{-1}}$, because the natural metric on the noncommutative torus is the metric g^{-1}, as we have seen above. Now in the above decoupling limit, the finiteness of this volume forces us to take also g to be small. And this is exactly

[1]We thank R. Dijkgraaf for pointing this out to us.

the limit in which we can make the identification of the noncommutative parameter θ with the B-field modulus b. It is also interesting to relate this to the theory on the classical torus. The metric there is the T-dual metric G. The dimensionfull volume of this metric is given by the expression $\ell_s^2\sqrt{\det g}/(b^2+\det g)$. In this paper we are interested in a situation where the B-field modulus b is finite (of order one). This now implies that this T-dual volume always becomes zero in the limit $\alpha' \to 0$. Therefore, the $\alpha' \to 0$ limit, where stringy effects should decouple, would then be described by the infrared, free theory. This is however not at all what we should expect. In the description using the noncommutative torus however, we see that we can still remain with a theory with finite (dimensionfull) in the limit where the stringy effects are decoupled.

4. Some notes on the noncommutative torus $\mathbb{T}^2_\theta$

In this section we briefly discuss the necessary ingredients from noncommutative geometry that we need. A quite extensive description of noncommutative geometry in general can ber found in Connes' book [25], where also the noncommutative two-torus is discussed in some detail. Some aspects, although in a slightly different form, were also discussed in [4]. The description of the trivial abelian bundle was discussed in detail in [5].

4.1 Bundles on the noncommutative two-torus

The noncommutative two-torus is defined through the C^*-algebra of functions, which are generated by Fourrier modes $U_i = \mathrm{e}^{2\pi i x^i}$, $i = 1, 2$. In contrast to the classical case, these functions do not commute, but the modes satisfy commutation relations

$$U_1U_2 = \mathrm{e}^{2\pi i\theta}\, U_2U_1\,, \qquad \text{or} \qquad \left[x^1, x^2\right] = \frac{\theta}{2\pi i}\,, \tag{4.1}$$

where θ is a real parameter called the noncommutative parameter. Furthermore, we need a set of derivations ∂_i. They are formally defined through their commutation relations with the coordinates, which are the usual ones. Note that they are straightforward generalizations of the derivatives, and commute among themselves.

Gauge bundles on the noncommutative torus are also defined through their sections. As these can always be multiplied (on the right) by functions on the torus, they form a (right-) *module* for the C^*-algebra of functions. This serves in noncommutative geometry as the definition of these bundles. Gauge bundles can be endowed with a connection ∇. Connections satisfy the same derivation requirement as in the commutative case

$$\nabla_i\,(\psi f) = (\nabla_i\,\psi)\, f + \psi\partial_i f\,. \tag{4.2}$$

This condition implies that the gauge connection also in the noncommutative case has to commute with the full algebra of functions. Therefore, it must be a function not of the coordinates x^i, but of the modified coordinates $\tilde{x}^i$ defined as

$$\tilde{x}^i = x^i + \frac{i\theta}{2\pi}\epsilon^{ij}\partial_j\,, \tag{4.3}$$

which commute with the coordinates x^i and therefore with the algebra of functions. Note that these form an algebra similar to the x^i, but with opposite parameter $-\theta$. Gauge bundles can be defined through translation operators defining periodic boundary conditions. They are of the form

$$T_i = \mathrm{e}^{\partial_i}\,\Omega_i^{-1}\,. \tag{4.4}$$

Here the derivative generates the translation, and the Ω_i are a set of local gauge transformations. Gauge transformations should, as they should act only fiberwise, be functions of the commuting modes $\tilde{x}^i$. This also guarantees that the gauge field satisfies the derivation condition (4.2) after a gauge transformation.

The trivial, abelian, gauge bundle has just the algebra of functions as its sections. The periodicity condition are simply determined by $T_i = \mathrm{e}^{\partial_i}$. The gauge connection can then be given by $\nabla_j = \partial_j + i\mathcal{A}_j(\tilde{x})$, where the gauge field has an expansion in the modes $\widetilde{U}_i = \mathrm{e}^{2\pi i\tilde{x}^i}$. Note that trivial $\mathrm{U}(N)$ bundles are also easily constructed like this, by simply letting $\mathcal{A}_j$ take values in the corresponding Lie-algebra.

We come now to the construction of non-abelian gauge bundles with gauge group $\mathrm{U}(N)$ and with nonzero magnetic flux M. We assume that N and M are relatively prime numbers. We can always take a tensor product with a trivial $\mathrm{U}(N_0)$-bundle to describe the general case. A $\mathrm{U}(N)$-bundle with flux M can be constructed following 't Hoofts approach, starting from $\mathrm{SU}(N)$ matrices satisfying

$$V_1V_2 = \mathrm{e}^{2\pi i\frac{M}{N}}\,V_2V_1\,. \tag{4.5}$$

They can be constructed as suitable 'clock and shift' matrices. The translation operators defining the periodicity conditions can then be given by

$$T_1 = \mathrm{e}^{\partial_1}\,V_1, \qquad T_2 = \mathrm{e}^{\partial_2}\,\mathrm{e}^{-\frac{2\pi iM}{N}\tilde{x}^1}\,V_2\,, \tag{4.6}$$

Here an abelian gauge transformation is added to assure that they commute.[2]

The gauge field should always commute with both the coordinates x^i and the translations T_i. This turns out to imply that they can be expanded in the modes

$$Z_1 = \mathrm{e}^{\frac{2\pi i}{N}\tilde{x}^1}\,V_2^{-K}\,, \qquad Z_2 = \mathrm{e}^{\frac{2\pi i}{N+M\theta}\tilde{x}^2}\,V_1^{K}\,, \tag{4.7}$$

where K is an integer satisfying $KM + NL = 1$ for some integer L. Note that the relative prime condition implies the existence of such an integer. These modes are

[2]This is necessary in order for a fundamental bundle to exist. In the string interpretation, we need fundamentals because string endpoints, which should be allowed for D-branes, are seen as fundamentals on the worldvolume of the D-brane.

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the non-abelian generalizations of the modes $\widetilde{U}_i$. They generate an algebra similar to that of the $\widetilde{U}_i$

$$Z_1 Z_2 = \mathrm{e}^{-2\pi i\hat{\theta}}\, Z_2 Z_1\,, \qquad \text{where} \qquad \hat{\theta} = \frac{K + L\theta}{N + M\theta}\,. \tag{4.8}$$

We can construct a linear covariant derivative by the ansatz

$$\exp \nabla_1^0 = T_1 Z_2^{-M}\,, \qquad \exp \nabla_2^0 = T_2 Z_1^M\,. \tag{4.9}$$

Here the exponent M is chosen such that the $\mathrm{SU}(N)$ matrices drop out, so that we can take the logarithm. The presence of the T_i guarantee the correct derivation property (4.2). The general connection can then written $\nabla_j = \nabla_j^0 + i\mathcal{A}_j(Z)$. From the linear connection above we can calculate the zero-mode of the field strength

$$F^0 = \frac{1}{2\pi i}\left[\nabla_1^0, \nabla_2^0\right] = \frac{M}{N + M\theta} I\,. \tag{4.10}$$

Note the different normalization of this field strength. This is a consequence of the fact that the canonical trace on this bundle is not normalized to N, but rather [25]

$$\mathrm{Tr}_\theta\, I = N + M\theta\,, \qquad \text{with} \qquad \int d^2x\, \mathrm{Tr}_\theta\, F = M\,. \tag{4.11}$$

Note that the magnetic flux is still an integer.

4.2 $\mathrm{SL}(2,\mathbb{Z})_N$ duality: Morita equivalence

we now give a short discussion of Morita equivalence, which is a mathematical equivalence for bundles on the noncommutative torus. In the application to string theory, this duality is directly related to T-duality.

We saw above that the modes Z_i of the non-abelian gauge field generate an algebra (4.8) of the same type as the abelian modes $\widetilde{U}_i$. Therefore, we can identify the modes Z_i with modes of an abelian gauge field. This identifies the non-abelian gauge field $\mathcal{A}$ with an abelian gauge field $\widehat{\mathcal{A}}$. The corresponding abelian gauge bundle lives on a different noncommutative torus. Comparing the commutation relations of the Z_i with those of the $\widetilde{U}_i$, we see that this should be a torus with noncommutative parameter $\hat{\theta}$. So we find that the *non-abelian* gauge field $\mathcal{A}$ on the non-commutative torus $\mathbb{T}^2_\theta$ can be identified with a *trivial abelian* gauge field $\widehat{\mathcal{A}}$ on the dual torus $\widehat{\mathbb{T}}^2_{\hat{\theta}}$. Note that two gauge theories are in fact related by an $\mathrm{SL}(2,\mathbb{Z})$ transformation, with matrix

$$\begin{pmatrix} L & K \\ M & N \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\,. \tag{4.12}$$

Indeed, the transformed noncommutative parameter $\hat{\theta}$ in (4.8) is related by the above matrix through standard fractional linear transformations. The pair $(N, -M)$ is mapped by this matrix to $(1, 0)$, which is the gauge data appropriate for the trivial abelian bundle. The only thing still missing to get a direct mapping between the gauge fields is the normalization of the fields. To find the correct normalization we look at the linear connection ∇^0. The dual linear connection is the trivial one $\widehat{\partial}$. Taking care of the scaling of the coordinates, which can be read off from the explicit form of the Z_i, we find the explicit relation between the linear connections

$$\widehat{\nabla}_1^0 = \hat{\partial}_1 = (N + M\theta)\,\nabla_1^0 + 2\pi i M x^2\,, \qquad \widehat{\nabla}_2^0 = \hat{\partial}_2 = (N + M\theta)\,\nabla_2^0\,. \tag{4.13}$$

This implies a scaling of the gauge field with a factor of $(N + M\theta)$. The abelian field strength is then easily found in terms of the non-abelian one

$$\widehat{F} = (N + M\theta)^2 F - M(N + M\theta) I\,. \tag{4.14}$$

Note that the linear shift absorbs the zero-mode F^0 of the non-abelian field strength, so that indeed the abelian gauge bundle is trivial.

We saw that we can always relate a non-abelian gauge field on the noncommutative torus with an abelian gauge field on a dual torus. We can turn this around, and relate the abelian gauge field to several non-abelian gauge fields on different tori. This correspondence defines an equivalence called *Morita equivalence*. It is an equivalence relation between bundles on noncommutative tori. We saw that in fact it is an isomorphism of the corresponding adjoint bundles, which are generated by the Z_i. The above equivalence between the non-abelian bundle and the abelian one can be generated by a unimodular matrix (4.12). It was already noted that this matrix acts on the parameter θ to find $\hat{\theta}$. Moreover, it transforms the vector of charges $(N, -M)$, which characterize the non-abelian bundle, into the vector $(1, 0)$, which then stands for the trivial abelian bundle. Combining several of these equivalences, this implies that any of these can be generated by an $\mathrm{SL}(2, \mathbb{Z})$ transformation, acting on the bundle data $(N, -M)$ in the doublet representation, and on the noncommutative parameter θ with fractional linear transformations. Explicitly, these transformations act as follows

$$\begin{aligned}
\mathcal{A} &\to (C\theta + D)\mathcal{A}\,,\\
F &\to (C\theta + D)^2 F - C(C\theta + D)\,,\\
\theta &\to \frac{A\theta + B}{C\theta + D},
\end{aligned}$$

for

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})\,.$$

We already noted that the canonical trace in a bundle on the noncommutative torus is given by the non-trivial expression $N + M\theta$. Also remember that the adjoint bundles in the dual situations can be identified. Therefore the trace in one representation also defines a trace in the dual one. Of course, a trace can always be multiplied by a constant. As the trace has a canonical normalization, where the magnetic field has an integer zero-mode, the factor with which the trace changes can be found. The easiest way is by considering the formula for the trace of the unit in (4.11). With the action of $\mathrm{SL}(2,\mathbb{Z})$ on θ and on the doublet $(N,-M)$, we find

$$\mathrm{Tr}_\theta \to (C\theta + D)^{-1}\,\mathrm{Tr}_\theta\ . \tag{4.15}$$

To finish the discussion about the duality, we consider the transformation of the electric field $E^i = \delta\mathcal{L}/\delta\dot{\mathcal{A}}_i$. The transformation can now most directly be read off from the invariance of the combination $\mathrm{Tr}_\theta\, E^i \dot{\mathcal{A}}_i$. It follows that E^i is invariant.

4.3 Quantization of the electric flux

We now study the periodicity of the gauge field. As the $\mathrm{U}(N)$ gauge field takes values in a compact space, there are certain periodicity conditions. Most important is the $\mathrm{U}(1)$ factor, which gives a periodicity $\mathcal{A}_i \to \mathcal{A}_i + 2\pi$. This periodicity is generated by the electric field operator $E^i = \delta/\delta\mathcal{A}_i$. On the commutative torus this shift is generated by a gauge transformation, with $\Omega = \exp 2\pi i x^i$. On the noncommutative torus this is not a local gauge transition function, as we saw above. The gauge transformation that is most close to this on the noncommutative torus is $\Omega = \exp 2\pi i \tilde{x}^i$. We find

$$\nabla_j \to \mathrm{e}^{\theta^{ik}\partial_k}\,\nabla_j\,\mathrm{e}^{-\theta^{ik}\partial_k} - 2\pi i \delta_{ij}\ . \tag{4.16}$$

Now the shift part is generated by the electric field zero mode $\int \mathrm{Tr}\, E^i$, while the translation part can be generated by the total momentum operator, $\int \mathrm{Tr}_\theta\, P_i$. The gauge transformation on the gauge field above is generated on the wave function by the operator

$$\exp\left(2\pi i \int \mathrm{Tr}_\theta (E^i + \theta^{ik} P_k)\right)\ . \tag{4.17}$$

As it is a true gauge transformation (Ω is single valued on the noncommutative torus), this operation should act trivial on the wave function. Therefore, the quantization of the electric flux is modified on the noncommutative torus to

$$\int \mathrm{Tr}_\theta\, E^i = n^i - \theta^{ij} m_j\,, \qquad \text{where} \qquad \int \mathrm{Tr}_\theta\, P_i = m_i\ . \tag{4.18}$$

Here both n^i and m_i are integers. Note that the total momentum is still quantized in the usual way, because the derivatives still commute.

5. Discussion and conclusion

We found that the Born-Infeld theory on the dual noncommutative torus gives the correct U-duality invariant BPS spectrum expected from string theory and M-theory.

This was also true for the theory on the commutative torus, when the constant B-field was included in the action in a different way. But when put on a noncommutative torus, it turn out that the map corresponding to the T-duality part of the duality group can be extended to an exact map between the different gauge theories. This map turns out to be given by Morita equivalence. This was already appreciated in [4], and generalized in [10] for the M(atrix) theory. We found however that this is only consistent with the identification of the B-field modulus when the volume σ_2 is zero. We argued that for finite values of this volume, there should be corrections to the noncommutative geometry, which enable us to identify the noncommutative parameter θ with the full complex modulus σ. However, as already mentioned, such a generalization is not known to us at present; although it is very natural that it should exist. On the other hand, we argued that the fact that the dimensionless volume is zero is related to the supression of stringy effects. So we expect that for a description at nonzero σ_2 we have to include also the stringy effects, which might modify the theory considerably. The gauge theory description then may not be sufficient.

The nontrivial $\mathrm{SL}(2,\mathbb{Z})_N$ duality group is generated by two operations. One is a shift transformation, which maps the magnetic flux M to $M+N$. This duality is quite trivial in the gauge theory. The other one, which interchanges the magnetic flux M and the rank N, is much more nontrivial, and is a two-dimensional version of the Nahm transformation, described in [26]. In the 'classical' description on the commutative torus, this duality is quite involved; it is formulated in terms of zero-modes of the Dirac operator. In the setting of noncommutative geometry however, this transformation is much simpler. As we saw, the adjoint fields on both sides can really be identified. This identification can even be done locally, as far as this can be said in noncommutative geometry.

The U-duality transformations that are not part of the T-duality group are still a bit mysterious. They are only seen at the level of the BPS spectrum in the gauge theory. In [26], it was noted that for Yang-Mills on the three-torus a subset of the BPS spectrum even has the correct U-duality invariant counting formula. This result can directly be reduced to two dimensions. Because we identified the Yang-Mills theory as a limit of the Born-Infeld theory, we expect also the counting of the corresponding BPS states in the Born-Infeld theory to be U-duality invariant.

Acknowledgments

It is a pleasure to thank Gijsbert Zwart, Feike Hacquebord, Robbert Dijkgraaf and Adriene Criscuolo for interesting discussions and helpful comments. C.H. is financially supported by the Stichting FOM. The research of E.V. is partly supported by the Pionier Programme of the Netherlands Organisation for Scientific Research (NWO).

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